# Assignment 13.

This homework is due *Thursday* April 26.

There are total 39 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

(1) (15.2.1abc) Express each of the rational numbers as finite simple continued fraction:
   (a) [2pt] $-19/51$,
   (b) [2pt] $187/57$,
   (c) [2pt] $71/55$.

(2) [2pt] (15.2.3) If $r = [a_0, a_1, \ldots, a_n]$, where $r > 1$, show that $\frac{1}{r} = [0, a_0, a_1, \ldots, a_n]$.

(3) [4pt] (15.2.8+) If $C_k = p_k/q_k$ is the $k$th convergent of the simple (finite or infinite) continued fraction $[a_0, a_1, a_2, \ldots]$, establish that

$$q_k \geq 2^{(k-1)/2} \qquad \text{for } 2 \leq k \leq n.$$

(*Hint:* Observe that $q_k = a_k q_{k-1} + q_{k-2} \geq 2q_{k-2}$.)

(4) (a) [2pt] Find $\alpha = [1, \overline{2}]$.
   (b) [2pt] Compute the 0th, 1st,..., 5th convergents of $\alpha$.
   (c) [2pt] Give a "quadratic" upper bound for the difference between $\alpha$ and the 5th convergent in (a).

(5) (15.3.1abd) Evaluate each of the following infinite simple continued fractions.
   (a) [2pt] $[\overline{2, 3}]$,
   (b) [2pt] $[0, \overline{1, 2, 3}]$,
   (c) [2pt] $[0, 1, \overline{3, 1}]$,

(6) (15.1.5)
   (a) [3pt] For any positive integer $n$, show that $\sqrt{n^2+1} = [n, \overline{2n}]$.
   (*Hint:* Integer part of $\sqrt{n^2+1}$ is $n$ because $(n+1)^2 > n^2 + 1$. Then notice that

$$n + \sqrt{n^2+1} = 2n + (\sqrt{n^2+1} - n) = 2n + \frac{1}{n + \sqrt{n^2+1}}.)$$

   (b) [3pt] For any positive integer $n$, show that $\sqrt{n^2+2} = [n, \overline{n, 2n}]$, $\sqrt{n^2+2n} = [n, \overline{1, 2n}]$.
   (c) [2pt] Use parts (a) or (b) to obtain the continued fraction of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{15}$ and $\sqrt{37}$.

(7) Given the infinite continued fraction $[1, 2, 3, 4, 5, \ldots]$, find the best rational approximation $a/b$ with denominator
   (a) [2pt] $b \leq 30$,
   (b) [2pt] $b \leq 157$

(8) [3pt] Given the infinite continued fraction $\alpha = [1, 3, 5, 5, 2, a_5, a_6, \ldots]$ find an interval with rational endpoints with length $\leq 1/5000$ such that for any choice of positive integers $a_5, a_6, \ldots$, the number $\alpha$ lies on that interval.